QUATERNIONS BRIEFLY

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Abstract. There's a strange prejudice that quaternions are needlessly complicated which dates back to the late 19th century. They really aren't: A few basic facts about them suffice to make them nearly as easy to use and understand as the two-dimensional complex numbers. Here's an account aimed at those having an acquaintance with the latter and a little linear algebra.

The numerical hyperspace of quaternions contains the plane of complex numbers, which contains the real line in turn. It has three "imaginary" dimensions instead of the complex plane's single one.[1] Just as complex numbers are real-coefficiented linear combinations of the form $a + ib$, where $i^2 = -1$, quaternions likewise have the form $a + ib + jc + kd$, where $i^2 = j^2 = k^2 = ijk = -1$ (which latter chain of equalities William Rowan Hamilton scratched on a Dublin bridge while out on a walk with his wife one fall day in 1843 after years of – among other projects – searching for a type of number that could represent space; and with which Eamon de Valera decorated his prison cell when the British government had him scheduled for execution) leads to the noncommutative multiplication table

$$\begin{array}{rrrr} 1 & i & j & k \\ i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1 \end{array}$$

from which one can compute the product of any two quaternions (assuming the distributive law, &c. [2]).

Quaternions that are actually just real numbers ($b = c = d = 0$) are called *scalar*. When the real component is zero instead, the quaternion is *imaginary*, or a *vector* (also called a *right* or *pure* quaternion). So each quaternion is the sum of a scalar and a 3-dimensional vector component. As with complex numbers, the conjugate operator $q \mapsto \bar{q} = a - ib - jc - kd$ negating the vector component of $q = a + ib + jc + kd$ cleans up the notation, letting one express q's magnitude $|q|$ as the square root of $q\bar{q}$ (always a nonnegative scalar), and the scalar and vector parts as

$$\mathrm{sc}\,(q) = \frac{q + \bar{q}}{2} \quad \text{and} \quad \mathrm{vc}\,(q) = \frac{q - \bar{q}}{2},$$

respectively.[3] (I'll often omit parentheses below to avoid clutter: $\mathrm{sc}\,q$ and $\mathrm{vc}\,q$.)

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Thanks to Walt Siemens for mentioning to me in 1978 that quaternionic $\sqrt{-1}$ is a sphere.

[1] *Dimension* here has the commonly-understood meaning. Among themslves, mathematicians generally qualify this word with the name of a field or division ring: *real dimension*, in this case. [Of course, the reader should look up unfamiliar terms, particularly when italicized.]

[2] assuming, in other words, that the quaternions form a *ring*

[3] This is an instance of *cartesian decomposition* in a *-algebra (pronounced, "*star-algebra*").

This customary use of "scalar" and "vector" instead of "real" and "imaginary" avoids confusion in important related contexts,[4] but is nonetheless problematic. In algebraic terms, the quaternions form a vector space over the real numbers, so *every* quaternion is really a vector, not just the purely imaginary ones. But this vector space actually contains its scalars as a subspace, leading people to attach the quality of being a vector to the complementary subspace.[5]

Negative one (-1) has more square roots than just i, j, k and their opposites. In fact the full set of them is the whole surface of the vectors' unit sphere (centered at zero): q^2 is -1 just when $\mathrm{sc}\,(q) = 0$ and $|q| = 1$. This *vector sphere* is the 2-dimensional equator of the 3-dimensional quaternion unit hypersphere (the set of all q for which $|q| = 1$, including those for which $\mathrm{sc}\,(q) \neq 0$) [6], regarding ± 1 as the hypersphere's north and south poles.

Each point u in the vector sphere is just as good as i, j, or k in combining with the real number line to form an algebraically complete[7] complex plane consisting of the sums $t + us$ for all pairs t and s of real numbers. Each such plane contains the real line and also contains the line connecting just two antipodal points $\pm u$ of the sphere. (These antipodal imaginary units are mutually inverse, *both additively and multiplicatively*.) Hamilton used the term, "versor",[8] both for a point on the unit hypersphere in general and for $\widehat{q} = q/|q|$, given $q \neq 0$.[9] One may then view q as a complex number $\mathrm{sc}\,(q) + \widehat{\mathrm{vc}\,q}|\mathrm{vc}\,q|$, bearing in mind that it's distinct, as a quaternion, from $\mathrm{sc}\,(q) + i|\mathrm{vc}\,q|$ (except for the cases $\widehat{\mathrm{vc}\,q} = i$ and $\widehat{\mathrm{vc}\,q} = \mathrm{vc}\,q = 0$). $\widehat{\mathrm{vc}\,q}$, which I'll also write $\mathrm{vrt}\,q$, is termed the "right versor" of q.

Although quaternions generally don't commute multiplicatively, certain pairs do. Specifically, two quaternions will commute when (and only when) they're in the same complex plane, or equivalently when a product (or a ratio) of their vector parts is scalar. In particular, q commutes with $\bar{q}$, $1/q = \bar{q}/(|q|^2)$, q^2, & $\mathrm{vc}\,q$.

For the cases in which quaternions don't commute, one must distinguish right division from left. I write

$$p/q = \left.\frac{p}{q}\right\rfloor \quad \text{and} \quad q\backslash p = \left\lfloor\frac{p}{q}\right.$$

for the y and x, respectively, such that $yq = p = qx$.

Like complex numbers, quaternions have a *polar decomposition* in terms of quaternion exponentials and logarithms. The exponential and logarithm of a quaternion q are

$$e^q = e^{\mathrm{sc}\,q}(\cos|\mathrm{vc}\,q| + \widehat{\mathrm{vc}\,q}\ \sin|\mathrm{vc}\,q|)$$

and

$$\ln q = \ln|q| + \widehat{\mathrm{vc}\,q}\ \arccos(\mathrm{sc}\,q/|q|).$$

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[4] *biquaternions* and the representation of quaternions as 2×2 matrices of complex numbers

[5] Ironically, the term, "scalar," connotes total order, yet its utility in the context of general vector spaces is in generalizing the coefficient domain from the ordered real line to number spaces of higher dimension that are not so ordered. The relative character of *scalar* and *vector* is brought out by the notion of a *division-ring module* over the quaternions.

[6] The unit hypersphere $\{q \in \mathbb{H} : q\bar{q} = 1 = \bar{q}q\}$ is a multplicative group, an instance of the *unitary* group in a *-algebra. ($\mathbb{H}$ symbolizes the quaternions, conflictingly also hyperbolic space.)

[7] The plane will contain n (possibly coinciding) solutions of each n^{th}-degree polynomial equation whose coefficients are points in this plane, where n is a whole number.

[8] coining "tensor" alongside it for $|q|$, a usage now obsolete.

[9] With the additional prescription $\widehat{0} = 0$, $\widehat{()}$ generalizes the *signum* function $\mathrm{sgn}\,()$ defined likewise for real numbers, and extends moreover to inner-product spaces generally.

q's polar decomposition is then

$$q = |q|e^{u\phi}$$

where $u = \operatorname{vrt} q$ and $\phi = (\operatorname{vc} \ln q)/u = \arccos(\operatorname{sc} q/|q|)$. The latter is sometimes called the *argument* of q; I'll write it $\arg(q) = \arg q$. It and accordingly $\ln q$ have a countable infinitude[10] of values. Note that $\operatorname{vrt} e^{u\phi} = -u$ when $\sin\phi < 0$; more generally, it's $u \operatorname{sgn} \sin\phi$.

One can use a single quaternion instead of the 3×3 matrix of nine real numbers that more naïvely represents spatial orientation, ærospace *attitude*.

Said naïve representation envisions two rectangular cartesian coördinate systems with a common origin at an aircraft's center of mass (say), a common distance unit along all axes, such that one coördinate system is fixed to the aircraft (assumed rigid) while the other coördinate system has a fixed attitude. For any given attitude of the aircraft, every point's aircraft-fixed coördinate triple can be converted to the same point's coördinate triple in the fixed-attitude system by multiplying by the 3×3 matrix of direction cosines of the respective orthonormal basis vectors, which thus corresponds to the attitude and represents it.

The determinant of such a matrix is ± 1. When it's positive, the coördinate transformation is called a *rotation*. When also the matrix itself isn't just the identity matrix, the set of all points whose coördinates are the same in both systems is a line, called the rotation's *axis*, passing through the origin, and there's a nonzero angle θ, unique within the range $[0, \pi]$, such that for every point not in the axis, the aircraft-fixed and fixed-attitude coördinates x & y of the point differ from the coördinates z of the nearest point on the axis, common to both coördinate systems, by triples whose direction cosine $\widehat{x - z} \cdot \widehat{y - z}$ is $\cos\theta$.

Regarding triples representing a point as vector quaternions, one may effect the same aircraft-fixed to fixed-attitude transformation of the point's position vector by both left-multiplying and right-dividing (it doesn't matter which operation is first because quaternion multiplication is associative)[11] by either versor $q = \pm e^{u\theta/2}$, *i.e.* $y = qx/q = qx\overline{q}$, where u is that one (the same for all off-axis points) of the vector sphere's two rotation axis intercepts for which $0 < \det(u, x, c)[= -\operatorname{sc}(u \operatorname{vc}(xc))]$, where[12] $g = \pm e^{u\theta/4}$ & $c = gx\overline{g}$.

The two antipodal versors q (and no others) each correspond thus to the attitude and likewise represent it.

References

1. John Conway & Derek Smith, *On Quaternions and Octonions*, **2003**; Wellesley, A.K. Peters
2. Wikipedia, "Classical Hamiltonian Quaternions", **2014**; http://en.wikipedia.org/wiki/Classical_Hamiltonian_quaternions
3. M.J. MacManus, *Eamon de Valera*, **1946**; https://archive.org/details/eamondevalera007589mbp

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[10] meaning that of these values, there's a distinct one for each whole number and *vice versa*.

[11] Mathematicians use the term, *conjugation*, for this compound of multiplication and division on opposite sides by the same value, particularly appropriate in the case of algebraically unitary operators such as versors wherewith divison is precisely multiplication by the conjugate.

[12] So $y = gc\overline{g}$. The simpler $0 < \det(u, x, y)$ would serve instead but for the case $\theta = \pi$.